Dynamic Programming

Econ 501

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1 Outline

- Sequential Problem

- Functional Equation

- Principle of Optimality

- Stochastic Version

- References

2 Sequential Problem

$$Max_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \qquad \text{(SP)}$$

$$\begin{aligned} \text{s.t.} \qquad x_{t+1} &\in \Gamma(x_t) \qquad \forall t \\ x_0 &\in X \ \text{ given} \end{aligned}$$

Examples:

- One sector model

$$Max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(c_t) \tag{1}$$

$$\begin{aligned} \text{s.t.} \quad (k_{t+1} + c_t, k_t) &\in Y \quad\quad Y \subseteq \mathbb{R}^2, \text{or } k_{t+1} + c_t \leq f(k_t) \\ k_0 &\in \mathbb{R} \text{ given} \end{aligned}$$

- Multiple sector

$$Max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(c_t) \tag{2}$$

$$\begin{aligned} \text{s.t.} \quad (k_{t+1} + c_t, k_t) &\in Y \quad\quad Y \subseteq \mathbb{R}^{2l} \\ k_0 &\in \mathbb{R}^l \text{ given} \end{aligned}$$

3 Functional Equation

or recursive formulation: the general structure of the decision problem recurs every period.

Bellman Equation

$$\begin{aligned} W(x) &= Sup_{\ x' \in \Gamma(x)} \left\{ F(x, x') + \beta W(x') \right\} \qquad \text{(FE)} \\ x_0 &\in X \ \text{ given, } \Gamma : X \to X \end{aligned}$$

4 Some Definitions

- The value function

$$
\begin{aligned}
V &: \mathbb{R}^l \rightarrow \mathbb{R} \\
V(x_0) &= Max_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1})
\end{aligned} \tag{3}
$$

- A plan

$$\{x_{t+1}\}_{t=0}^{\infty} \text{ in } X$$

- The set of feasible plans

$$\Pi(x_0) = \left\{ \{x_{t+1}\}_{t=0}^{\infty} : x_{t+1} \in \Gamma(x_t), \quad \forall t \right\}$$

5 Assumptions

- $\Gamma(x)$ non empty

- $\forall x_0 \in X$ and $\widetilde{x} \in \Pi(x_0)$, $\lim_{n\to\infty} \sum_{t=0}^{n} \beta^t F(x_t, x_{t+1})$ exists.

6 Principle of Optimality

1. The value function defined in (5) satisfies the Bellman Equation, hence

   If $|V(x_0)| < \infty$, then

   $$V(x_0) \geq F(x_0, x') + \beta V(x') \qquad \forall x' \in \Gamma(x_0)$$

   and for any $\varepsilon > 0$

   $$V(x_0) \leq F(x_0, x') + \beta V(x') + \varepsilon \qquad \text{some } \; x' \in \Gamma(x_0)$$

   If $V(x_0) = +\infty$, then there exist a sequence $\left\{x'_k\right\} \in \Gamma(x_0)$ such that

   $$\lim_{k\to\infty} F(x_0, x'_k) + \beta V(x'_k) = +\infty$$

   If $V(x_0) = -\infty$, then $\forall x' \in \Gamma(x_0)$ such that

   $$F(x_0, x') + \beta V(x') = -\infty \quad \forall x' \in \Gamma(x_0)$$

2. If a sequence $\left\{x_{t+1}^*\right\}_{t=0}^{\infty}$ is a solution to (1) then the sequence satisfies the Bellman Equation

$$V(x_t^*) = F(x_t^*, x_{t+1}^*) + \beta V(x_{t+1}^*)$$

3. Let W^* be a solution to (FE) and $\lim_{n\to\infty} \beta^n W^*(x_n) = 0 \ \forall \ \widetilde{x} \in \Pi(x_0) \forall x_0 \in X$, then

$$W^* = V$$

4. If a feasible plan $\widetilde{x}^* \in \Pi(x_0)$ satisfies,

$$V(x_t^*) = F(x_t^*, x_{t+1}^*) + \beta V(x_{t+1}^*) \qquad \forall t$$

and $\lim_{n\to\infty} \beta^n V(x_n^*) = 0$, then x is a solution to (1)

Sketch of the Proof: in class...

7 Stochastic Version

7.1 Sequential Problem

$$Max_{\{x_{t+1}\}_{t=0}^{\infty}} E\left[\sum_{t=0}^{\infty} \beta^t F(x_t, z_t, x_{t+1})\right] \qquad \text{(SP)}$$

$$\begin{aligned} \text{s.t.} \quad x_{t+1} &\in \Gamma(x_t, z_t) \qquad \forall t \\ x_0 &\in X \quad z_0 \in Z \qquad \text{given} \end{aligned}$$

where X is the set of possible values for the endogeneous state variable and Z is that for the exogenous state space variable. (X, Υ) and (Z, Ξ) are measurable spaces and $(S, \Theta) = (X \text{x} Z, \Upsilon \text{x} \Xi)$ is the set of possible states of the system.

Shocks evolve according to a stationary transition function Q on (Z, Ξ) and the expectations is taken with respect to this meassure.

Information in this set up will be summarized by sequences $z^t = (z_1, z_2, ...) \in Z^t$ where (Z^t, Ξ^t) is a product space.

Example:

- One sector model

$$Max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} E \sum_{t=0}^{\infty} \beta^t U(c_t) \tag{4}$$

$$\begin{aligned} \text{s.t.} \quad (k_{t+1} + c_t, k_t) &\in Y(z_t) \quad Y(z_t) \subseteq \mathbb{R}^2, \text{or } k_{t+1} + c_t \leq z_t f(k_t) \\ k_0 &\in \mathbb{R} \; z_0 \in Z \text{ given} \end{aligned}$$

7.2 Functional Equation

Bellman Equation

$$\begin{aligned} W(s) &= W(x,z) = Sup_{\ x' \in \Gamma(x,z)}\ F(x,z,x') + \beta E\left[W(x',z')\right] \quad \text{(FE)} \\ x_0 &\in X \quad z_0 \in Z \ \text{ given, } \Gamma : X \text{x} Z \to X \end{aligned}$$

where

$$E\left[W(x',z')\right] = \int W(x',z')Q(dz',z)$$

If there exist a function W^* satisfying the bellman equation, then we can define the associated policy correspondence

$$G(s) = \left\{x'\right\} \in \Gamma(s) : W(s) = F(x,z,x') + \beta E\left[W(x',z')\right]$$

7.3 Some Definitions

- The value function

$$\begin{aligned} V &: \mathbb{R}^l \times Z \to \mathbb{R} \\ V(x_0, z_0) &= Max_{\{x_{t+1}\}_{t=0}^{\infty}} E\left[\sum_{t=0}^{\infty} \beta^t F(x_t, z_t, x_{t+1})\right] \end{aligned} \quad (5)$$

- A plan is a value $\pi_0 \in X$ and a sequence of measruble functions $\pi_t : Z^t \to X, \quad t = 1, 2, ...$

- The set of feasible plans π from $s_0 \in S$

$$\begin{aligned} \pi_0 &\in \Gamma(x_0, z_0) \\ \pi_t(z^t) &\in \Gamma(\pi_{t-1}(z^{t-1}), z_t) \quad \forall z_t \in Z^t,\ t = 1, 2, ... \end{aligned}$$

7.4 Assumptions

- $\Gamma(x,z)$ non empty-valued and the graph of Γ is ΥxΥxΞ measurable. Γ has a measurable selection,

$$\exists \quad h : S \rightarrow X \quad \text{such that } h(s) \in \Gamma(s), \text{all } s \in S$$

- $F : graph(\Gamma) \rightarrow \mathbb{R}$, is ΥxΥxΞ measurable and for each $s_0 \in S$ and $\pi \in \Pi(s_0)$,

$$F(\pi_{t-1}(z^{t-1}), z_{t,}\pi_t(z^t)) \text{ is } \mu^t(z_{0,}.) \text{ integrable}$$

and

$$\lim_{n \rightarrow \infty} \sum_{t=0}^{\infty} \beta^t \int F(\pi_{t-1}(z^{t-1}), z_{t,}\pi_t(z^t))\mu^t(z_{0,}dz^t) \text{ exist}$$

where if $B = A_1 x A_2 x ..... \in \Xi^t$

$$\mu^t(z_0, B) = \int_{A_1} ... \int_{A_{t-1}} \int_{A_t} Q(z_{t-1}, dz_t) Q(z_{t-2}, dz_{t-1}) ...... Q(z_0, dz_1)$$

7.5 Principle of Optimality

1. Let W^* be a solution to (5) and $\lim_{t\to\infty} \int_{z^t} W^*(\pi_{t-1}(z^{t-1}), z_t)\mu^t(z_{0,} dz^t)$ $= 0\ \forall\ \pi \in \Pi(s_0) \forall s_0 \in S$, then

$$W^* = V$$

2. Let G defined as before, and suppose G is non-empty and admits a measurable selection, then any plan π^*generated by G attains the supremum of the sequential problem.

Sketch of the Proof: see Stokey & Lucas

8 References

- Stockey and Lucas (1989). Recursive Methods in Economic Dynamics, Sections 2.1, 4.1-4.2 and 9.1

- Sargent and Ljungqvist (2004). Recursive Macroeconomic Theory, Section 3.1.