Optimal Control Problems

Econ 501

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1 Outline

- Calculus of Variations
  1. Euler- Lagrange Equation
  2. Dynamical Systems Hamiltonian
- Control Theory
  1. Pontryagin Maximum Principle
- Applications (in class)
- References

2 Calculus of Variations

2.1 Euler-Lagrange Equation

Basic problem is to find a curve $x^*(t) : [0, T] \rightarrow \mathbb{R}^*$ that minimizes the functional

$$I[x(.)] = \int_0^T L(x(t), \dot{x}(t))dt$$

among all functions $x()$ that satisfy $x(0) = x_0$ and $x(T) = x_1$, where L(x,v) is called the Lagrangian.

*potentially $\mathbb{R}^n$ but I would like to keep things simple to get the intuitions

Theorem Let $x^*(t)$ solve the calculus of variations problem. Then it solves the Euler-Lagrange differential equation:

$$\frac{d}{dt} L_v(x^*(t), \dot{x}^*(t)) = L_x(x^*(t), \dot{x}^*(t) \tag{EL}$$

Proof: In class

2.2 Dynamical Systems Hamiltonian

For any given curve $x(.)$ define

$$\lambda(t) = L_v(x(t), \dot{x}(t))$$

We need to assume that for all x, λ we can solve

$$\lambda = L_v(x, v)$$

for $v = v(\lambda, x)$

Define the Hamiltonian as

$$H(x, \lambda) = \lambda.v(x, \lambda) - L(x, v(x, \lambda))$$

Theorem Let $x^*(t)$ solve the (EL) equation and define $\lambda(t)$ as before. Then the pair $(x(.), \lambda(.))$ solves the Hamilston's euqation:

$$\begin{aligned} \dot{x}(t) &= H_\lambda(x(t), \lambda(t)) \\ \dot{\lambda}(t) &= H_x(x(t), \lambda(t)) \end{aligned} \tag{H}$$

Proof: In class

3 Control Theory

3.1 Pontryagin Maximum Principle

The basic problem here is to find a control $a^*(.) \in A$ some set of admissible controls, to solve for

$$P(a^*(.)) = \max_{a \in A} \int_0^T r(x(t), a(t))dt + g(x(T))$$

s.t.

$$\begin{aligned} \dot{x}(t) &= f(x(t), a(t)) \\ x(0) &= x_0 \end{aligned}$$

where $g(x(T))$ is some terminal payoff.

Then we can define the "control theory" Hamiltonian as

$$H(x, \lambda, a) = f(x, a).\lambda + r(x, a)$$

Theorem Assume $a^*(t)$ is optimal for the ODE and $x^*(t)$ is the corresponding trajectory. Then there exists a function $\lambda^*(t)$ such that

solve the (EL) equation and define $\lambda(t)$ as before. Then the pair $(x(.), \lambda(.))$ solves the Hamilston's euqation:

$$\begin{aligned} \dot{x}^*(t) &= H_\lambda(x^*(t), \lambda^*(t), a^*(t)) \\ \dot{\lambda}^*(t) &= -H_x(x^*(t), \lambda^*(t), a^*(t)) \end{aligned}$$

and

$$H(x^*(t), \lambda^*(t), a^*(t)) = \max_{a \in A} H(x^*(t), \lambda^*(t), a(t))$$

$$\lambda^*(T) = g_x(x(T))$$

4 References

Long, L (1992), "Optimal Control Theory & Static Optimization in Economics"

Evans, L (Version 0.2) "An Introduction to Mathematical Optimal Control Theory"